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Actuator and Exciter Placement for Flexible Structures

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Systematic approaches to select the optimal candidate sets for actuator and exciter placement are developed. The performance of control and system identification for flexible structures depends strongly on the locations of actuators and exciters. The proposed methods of optimal actuator and exciter placement rely on quantitative measures of modal degree of controllability and modal degree of excitability, respectively. Whereas modal degree of controllability is related to the minimum control input energy needed to regulate the system from initial modal disturbances, modal degree of excitability is related to the steady-state energy stored in each mode when white noise is applied to excite the system. The modal degrees of controllability and excitability offer the control system designer tools that allow the ranking of the effectiveness of a specific distribution of actuators and exciters and, hence, the choice of their locations on a rational basis. The effectiveness of the proposed measures is demonstrated by both computer simulation and an experiment.

Nomenclature

$\boldsymbol{A}$	= system matrix
$\boldsymbol{b}$	= input matrix
$\boldsymbol{b}'$	= control influence matrix
$\boldsymbol{D}$	= damping matrix
J_c	= actuator placement criterion
J_e	= exciter placement criterion
$\boldsymbol{K}$	= nonnegative definite stiffness matrix
$\boldsymbol{M}$	= positive definite inertia matrix
m	= number of actuators in a candidate set
n	= order of system
n_{act}	= number of locations possible for actuator placement, equal to number of locations in the full set of actuators
n_{excit}	= number of locations possible for exciter placement, equal to number of locations in the full set of exciters
$\boldsymbol{q}(t)$	= $n \times 1$ general displacement vector
$\boldsymbol{u}(t)$	= $m \times 1$ control input vector
$\boldsymbol{x}(t)$	= $2n \times 1$ state vector
$\varepsilon()$	= energy of ( )
$\boldsymbol{\eta}(t)$	= $n \times 1$ modal displacement vector
ς_i	= damping ratio
Ψ	= structural mode shape matrix
ω_{di}	= damped natural frequency
ω_i	= undamped natural frequency

Introduction

TWO issues concerning the control and identification of large flexible structural systems are addressed. One is how to select the locations of actuators to control the system, and the other is how to select those of exciters to perform modal testing.

There have been various attempts to develop a systematic framework for selecting an optimal candidate set for actuators.[1–4] It is known that certain performance goals cannot be achieved through feedback control, regardless of the control law used, if a controllability criterion is not met. Actuators should be placed at locations where the controllability of the system is guaranteed. However, the standard tests of controllability cannot be used directly in optimal actuator placement because they can provide only binary information. Various definitions of degree of controllability have been developed to evaluate the quantitative performance of a system with actuators at specific locations. Müller and Weber[1] defined three physical measures of controllability that are related to eigenvalues of the characteristic matrix of the system. Viswanathan et al.[2] presented a definition of the degree of controllability based on the concept of the recovery region, which is a set of initial conditions that can be returned to an origin in time T under input saturation constraint $|\boldsymbol{u}_i(t)| \leq 1$. Hamdan and Nayfeh[3] extended the concept of the Popov–Belevitch–Hautus eigenvector test[5] to define a degree of controllability for a specific mode, and Lim[4] introduced a cost function that is the weighted projection of structural modes into the intersection subspace of the controllable and observable subspaces.

We propose a novel concept titled modal degree of controllability (MDOC), which represents the relative performance of a specific candidate set with a predetermined number of actuators compared to the performance achievable with the full set of actuators. The full set of actuators indicates the case when the actuators are installed at all of the available candidate locations. Of course, we cannot afford the full set of actuators, and it will be used for comparison purpose only. The performance metric used in the definition is the control energy required to regulate the system from a disturbance of a specific structural mode. The cost function, which is the weighted sum of MDOCs corresponding to the modes of interest, can be used to find optimal actuator locations. By placing a predetermined number of actuators at the candidate set that optimizes the cost function, we can minimize the control energy required to regulate the system from disturbances that have several modal components.

The selection of exciter location for modal testing is also addressed. We may think of using the same method developed in the actuator placement problem for exciter placement as well. However, applying a control-oriented criterion to the exciter placement problem is not appropriate because actuators and exciters have different objectives: actuators are used to regulate systems excited by an external disturbance, whereas exciters are used to excite appropriate modes for the identification of unknown systems. We propose a system-identification-oriented modal degree of excitability (MDOE) that enables us to select exciter locations systematically. Compared with MDOC, MDOE indicates the relative performance of the selected candidate set for distributing the steady-state energy to a specific mode. By using a cost function that consists of MDOEs, we can place exciters so that the steady-state energy distribution among the modes of interest may be optimized.

It will also be shown that, if the frequencies of the structure are widely spaced and damping is light, MDOC and MDOE have the same values, even though they are defined based on different theoretical and physical backgrounds. This means that the optimal candidate sets for actuators and exciters coincide for a single mode under the specified condition. However, the optimal candidate sets will not be identical due to the difference in their cost functions, in the case that multiple modes are of interest.

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System Modeling

The mathematical model of a linear, time-invariant (LTI) flexible structure is given by

$$M\ddot{q}(t) + D\dot{q}(t) + Kq(t) = b'u(t) \tag{1}$$

where $b'(\in \Re^{n\times m})$ is the control influence matrix, which has columns chosen from those of the $n \times n_{\text{act}}$ matrix B'. Each column in matrix B' represents an available location and direction of an actuator or an exciter.

By coordinate transformation, Eq. (1) can be changed into the following modal form:

$$\ddot{\eta}(t) + \text{diag}(2\varsigma_i\omega_i)\dot{\eta}(t) + \text{diag}\left(\omega_i^2\right)\eta(t) = \Psi^T b' u(t) \tag{2}$$

Equation (2) can be rewritten in state-space form as follows:

$$\dot{x}(t) = Ax(t) + bu(t) \tag{3}$$

where

$$x(t) = \{\omega_1\eta_1 \quad \dot{\eta}_1 \quad \cdots \quad \omega_n\eta_n \quad \dot{\eta}_n\}^T \tag{4}$$

$$A = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_n \end{bmatrix} \qquad \left(A_i = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & -2\varsigma_i\omega_i \end{bmatrix}\right) \tag{5}$$

$$b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \qquad \left(b_i = \begin{bmatrix} 0 & \cdots & 0 \\ b_{1i} & \cdots & b_{mi} \end{bmatrix}\right) \tag{6}$$

where $\{b_{1i} \quad \cdots \quad b_{mi}\}$ is the ith row vector of $\Psi^T b'$. If m in Eq. (6) turns out to be n_{act}, then b becomes a $2n \times n_{\text{act}}$ matrix B, where

$$B = \begin{bmatrix} 0 \\ B' \end{bmatrix}$$

MDOC

From now on, we define the control objective as regulating a system from initial disturbances x_0 to zero within a given time interval of $t_1 - t_0$, with the minimum control energy $E(t_0, t_1; x_0)$ defined as follows[1]:

$$E(t_0, t_1; x_0) = \underset{u}{\text{minimum}} \int_{t_0}^{t_1} \|u(\tau)\|^2 \, d\tau$$

$$x(t_0) = x_0, \quad x(t_1) = 0, \quad [t_0, t_1] \text{ fixed} \tag{7}$$

If the system in Eq. (5) is controllable, then the optimal solution to Eq. (7) is given as follows[1]:

$$E(t_0, t_1; x_0) = x_0^T W_c^{-1}(t_1, t_0) x_0 \tag{8}$$

where

$$W_c(t_1, t_0) = \int_{t_0}^{t_1} e^{A(t_0-\tau)} b b^T e^{A^T(t_0-\tau)} \, d\tau$$

is the controllability grammian matrix[1] of (A, b). If we assume that system matrix A is fixed, then $E(t_0, t_1; x_0)$ is a function of the time interval $t_1 - t_0$ and the input matrix b. For a fixed time interval, the magnitude of $E(t_0, t_1; x_0)$ will vary as we change actuator locations and, correspondingly, as the input matrix b is changed. If $E(t_0, t_1; x_0)$ of system Σ_1 is smaller than that of system Σ_2, system Σ_1 can be said to be more controllable than system Σ_2. $E(t_0, t_1; x_0)$ provides a quantitative measure of controllability, but this measure depends on the initial condition x_0. We assume that a set of important modes is known through a priori analyses, and we want to determine the controllability of the system for each mode of interest.

The minimum control energy required to regulate the system from a specific modal disturbance is written as $E(t_0, t_1; x_0^i)$, where x_0^i is the modal vector defined as

$$x_0^i \equiv a_{i1}\{0 \ \cdots \ 0 \ \ 1 \ \ 0 \ \cdots\}^T + a_{i2}\{0 \ \cdots \ 0 \ \ 1 \ \ 0 \ \cdots\}^T$$

$$(2i-1)\text{th} \qquad\qquad 2i\text{th} \tag{9}$$

$$\omega_i\eta_i \qquad\qquad \dot{\eta}_i$$

and a_{i1} and a_{i2} are arbitrary constants representing the displacement and velocity component of a specific mode. Depending on a_{i1} and a_{i2}, there is an infinite number of initial conditions that can excite the system with a corresponding specific mode. Therefore, we require initial condition to disturb the system with a unit energy and find the maximum of $E(t_0, t_1; x_0^i)$ computed for all possible initial conditions:

$$\max_{\varepsilon(x_0^i)=1} E\left(t_0, t_1; x_0^i\right) = \max_{\varepsilon(x_0^i)=1} \left[\left(x_0^i\right)^T W_c^{-1}(t_1, t_0) x_0^i\right] \tag{10}$$

Because the energy in the system is equivalent to the 2-norm of its state vector in modal coordinates, x_0^i is constrained such that

$$\left\|x_0^i\right\|_2 = 1 \tag{11}$$

which means that a_{i1} and a_{i2} in Eq. (9) will satisfy the following constraint:

$$a_{i1}^2 + a_{i2}^2 = 1 \tag{12}$$

Under the constraint just given, the optimizing pair (a_{i1}^*, a_{i2}^*) that maximizes $E(t_0, t_1; x_0^i)$ changes both with time interval and with modal properties. If we define ξ_i as the x_0^i with the optimizing pair (a_{i1}^*, a_{i2}^*), then

$$\max_{\|x_0^i\|_2=1} E\left(t_0, t_1; x_0^i\right) = E(t_0, t_1; \xi_i) \tag{13}$$

In summary, $E(t_0, t_1; \xi_i)$ represents the maximum of the minimum control energy required to regulate the system from its modal disturbance with unit initial energy. But $E(t_0, t_1; \xi_i)$ converges to zero rapidly as the time interval $t_1 - t_0$ increases. This makes it difficult to use $E(t_0, t_1; \xi_i)$ for comparison purposes. To make $E(t_0, t_1; \xi_i)$ more convenient to use, we define the degree of controllability for a specific mode, i.e., MDOC, as

$$\text{MDOC}_i \equiv \frac{\bar{E}(t_0, t_1; \xi_i)}{E(t_0, t_1; \xi_i)} = \frac{\xi_i^T \bar{W}_c^{-1}(t_1, t_0)\xi_i}{\xi_i^T W_c^{-1}(t_1, t_0)\xi_i} \tag{14}$$

where $\bar{E}(t_0, t_1; \xi_i)$ is the maximum of minimum input energy when the actuators are located at all of the possible locations, i.e., the full set of actuators is used. $W_c(t_1, t_0)$ and $\bar{W}_c(t_1, t_0)$ in Eq. (14) are the controllability grammian matrices defined for input matrix b and B, respectively. From Eq. (14), MDOC can be interpreted as the relative regulation performance of a candidate actuator set for a specific mode.

MDOC has the following characteristics.

1) The bound of MDOC is given as

$$0 < \text{MDOC}_i \le 1\left[\because \xi_i^T W_c(t_1, t_0)^{-1}\xi_i \ge \xi_i^T \bar{W}_c(t_1, t_0)^{-1}\xi_i\right]$$

Thus, MDOC can be used not only for comparing the regulation performance of a specific candidate set but also for observing the effect of the number of actuators on modal controllability. The latter fact can be useful in choosing the number of actuators to be used. If the MDOC for a specific number of actuators is small, it implies that an increase in the number of actuators can improve the controllability of the mode.

2) MDOC is invariant for LTI systems. Whereas $E(t_0, t_1; \xi_i)$ converges to zero as the duration time increases, $\bar{E}(t_0, t_1; \xi_i)/E(t_0, t_1; \xi_i)$ takes on a constant value, which means that the relative regulation performance of a candidate set is independent of the control time interval and becomes a characteristic of an LTI system.

If initial disturbance is given in a linear combination of selected modes of interest, the maximum of the minimum control energy required to regulate the system from the disturbance will be written as

$$\sum_i^{\text{selected}} \alpha_i \boldsymbol{E}(t_0, t_1; \boldsymbol{\xi}_i)$$

where α_i represents the relative importance of the ith mode and can be taken to be proportional to its contribution to the system response. Consistent with the definition of MDOC, the actuator placement criterion for selected modes of interest can be proposed so as to minimize the following cost function:

$$J_c = \frac{\sum_i^{\text{selected}} \alpha_i \boldsymbol{E}(t_0, t_1; \boldsymbol{\xi}_i)}{\sum_i^{\text{selected}} \alpha_i \bar{\boldsymbol{E}}(t_0, t_1; \boldsymbol{\xi}_i)} \tag{15}$$

Thus, the cost function represents a ratio of the minimum control energy of a specific candidate set with respect to that of the full set.

Equation (15) can be written as

$$J_c = \sum_i^{\text{selected}} \left[\frac{\alpha_i \boldsymbol{E}(t_0, t_1; \boldsymbol{\xi}_i)}{\sum_i^{\text{selected}} \alpha_i \bar{\boldsymbol{E}}(t_0, t_1; \boldsymbol{\xi}_i)} \right] \cdot \frac{\alpha_i \bar{\boldsymbol{E}}(t_0, t_1; \boldsymbol{\xi}_i)}{\alpha_i \bar{\boldsymbol{E}}(t_0, t_1; \boldsymbol{\xi}_i)}$$

$$= \sum_i^{\text{selected}} \left[\frac{\alpha_i \bar{\boldsymbol{E}}(t_0, t_1; \boldsymbol{\xi}_i)}{\sum_i^{\text{selected}} \alpha_i \bar{\boldsymbol{E}}(t_0, t_1; \boldsymbol{\xi}_i)} \right] \cdot \left[\frac{1}{\bar{\boldsymbol{E}}(t_0, t_1; \boldsymbol{\xi}_i) / \boldsymbol{E}(t_0, t_1; \boldsymbol{\xi}_i)} \right]$$

$$= \sum_i^{\text{selected}} \mu_i \frac{1}{\text{MDOC}_i} \tag{16}$$

where

$$\mu_i = \frac{\alpha_i \bar{\boldsymbol{E}}(t_0, t_1; \boldsymbol{\xi}_i)}{\sum_i^{\text{selected}} \alpha_i \bar{\boldsymbol{E}}(t_0, t_1; \boldsymbol{\xi}_i)} \qquad \left(\sum_i^{\text{selected}} \mu_i = 1 \right) \tag{17}$$

Equation (16) indicates that the cost function is a function of MDOCs of the selected modes of interest. Also note that larger MDOCs will make a smaller cost function, which corresponds to a better candidate set of actuators.

MDOE

Modal testing is usually used to validate the system model obtained by finite element analysis (FEA). FE model modification is repeated until frequency response functions given by two methods show the similar tendency to a satisfactory level. Exciter locations have significant effects on the accuracy of the modal testing, and they can be determined systematically if we use modal properties of the system obtained by FEA.

In spite of its importance, research on exciter placement has been relatively rare compared with research on the actuator placement.[6,7] The main concern of the latter part of this paper is how we should modify the methodology developed for actuator placement so that it can be used for exciter placement.

Whereas an actuator is used for control to counter external disturbances, an exciter is used for exciting the individual modes of an unknown system. Hence, the working conditions for the actuator and the exciter are different. Their differences can be summarized as follows. First, in the control problem, only the modes affected by a disturbance are important and need to be controlled, whereas all of the modes need to be excited evenly in system identification. Second, the input time history for control is usually determined by a feedback controller, whereas that for system identification is usually a random or sinusoidal input, which can be decided by the user. Third, the input energy required during the transient state is of main concern in control, whereas the input energy distribution at steady state is more important in system identification.

For the aforementioned reasons, we cannot use MDOC without modification for selection of exciter locations. Therefore, we will define MDOE for system identification. Assuming exciter inputs are mutually uncorrelated random signals of unit intensity such that

$$E[\boldsymbol{u}(t)\boldsymbol{u}(\tau)^T] = \boldsymbol{I}\delta(t-\tau) \tag{18}$$

where $E[\,]$ is the expected value, the steady-state covariance matrix of state vector $E[\boldsymbol{x}(t)\boldsymbol{x}(t)^T] = \boldsymbol{X}$ satisfies the following Lyapunov equation[7]:

$$\boldsymbol{AX} = \boldsymbol{XA}^T + \boldsymbol{bb}^T = \boldsymbol{0} \tag{19}$$

For state vector $\boldsymbol{x} = \{\omega_1\eta_1, \dot{\eta}_1, \ldots, \omega_n\eta_n, \dot{\eta}_n\}^T$, the ith block diagonal element of the solution of the Lyapunov equation will be

$$\boldsymbol{X}_i = E \begin{bmatrix} (\omega_i\eta_i)^2 & \omega_i\dot{\eta}_i\eta_i \\ \omega_i\dot{\eta}_i\eta_i & (\dot{\eta}_i)^2 \end{bmatrix} \tag{20}$$

Note that the trace of $\boldsymbol{X}_i$ is equal to the average value of the ith modal energy at steady state.[7] The MDOE using the property of $\text{tr}(\boldsymbol{X}_i)$ is defined as follows:

$$\text{MDOE}_i = \frac{\text{tr}(\boldsymbol{X}_i)}{\text{tr}(\bar{\boldsymbol{X}}_i)} \qquad (0 \le \text{MDOE}_i \le 1) \tag{21}$$

where $\boldsymbol{X}_i$ and $\bar{\boldsymbol{X}}_i$ are the ith block diagonal element of the solution of the Lyapunov equation when a specific candidate set and a full set of exciters are used, respectively. MDOE represents a ratio of the steady-state modal energy level obtainable by a specific candidate set of exciters with respect to the energy level with a full set of exciters. Consistent with the criterion for actuator placement, the criterion for exciter placement using MDOE can be defined to maximize

$$J_e = \frac{\sum_i^{\text{selected}} \beta_i \cdot \text{tr}(\boldsymbol{X}_i)}{\sum_i^{\text{selected}} \beta_i \cdot \text{tr}(\bar{\boldsymbol{X}}_i)} = \sum_i^{\text{selected}} \sigma_i \text{MDOE}_i \tag{22}$$

where

$$\sigma_i = \frac{\beta_i \text{tr}(\bar{\boldsymbol{X}}_i)}{\sum_i^{\text{selected}} \beta_i \text{tr}(\bar{\boldsymbol{X}}_i)} \qquad \left(\sum_{i=1}^{n} \sigma_i = 1 \right) \tag{23}$$

and β_i is the weighting factor for each mode that is decided by a designer. Note that, whereas in Eq. (16) the goal is to minimize the cost function, in Eq. (22) we want to maximize the cost function. This difference results from the fact that, whereas in the actuator placement criterion we use the control energy as a performance measure, which needs to be minimized, we use the steady-state energy, which needs to be maximized, in exciter placement criterion.

MDOC vs MDOE

We have considered methods to select the optimal candidate sets for actuators and exciters. For the calculation of MDOC and MDOE, a priori knowledge of the controllability grammian matrix and the solution of Lyapunov equation is required, respectively.

The controllability grammian matrix, by its definition, has the following property[8]:

$$\frac{\text{d}}{\text{d}t} \boldsymbol{W}_c(t_1, t) = \boldsymbol{AW}_c(t_1, t) + \boldsymbol{W}_c(t_1, t)\boldsymbol{A}^T - \boldsymbol{bb}^T \qquad (t \le t_1)$$

$$\boldsymbol{W}_c(t_1, t_1) = \boldsymbol{0} \tag{24}$$

If the system is open-loop stable, there exists an equilibrium solution satisfying the following Lyapunov equation:

$$\boldsymbol{AW}_c^0 + \boldsymbol{W}_c^0\boldsymbol{A}^T - \boldsymbol{bb}^T = \boldsymbol{0} \tag{25}$$

Note that, by comparing Eqs. (19) and (25), we can find

$$\boldsymbol{W}_c^0 = -\boldsymbol{X} \tag{26}$$

The following relationship holds between the controllability grammian matrix $\boldsymbol{W}_c(t_1, t_0)$ and its equilibrium solution $\boldsymbol{W}_c^0$ (Ref. 8):

$$\boldsymbol{W}_c(t_1, t_0) = \boldsymbol{W}_c^0 - e^{-\boldsymbol{A}(t_1 - t_0)}\boldsymbol{W}_c^0 e^{-\boldsymbol{A}^T(t_1 - t_0)} \tag{27}$$

We shall assume that the natural frequencies of the structure are widely spaced and that its damping is light. Then, in modal coordinates with state vector $\boldsymbol{x} = \{\omega_1\eta_1 \;\; \dot{\eta}_1 \;\; \cdots \;\; \omega_n\eta_n \;\; \dot{\eta}_n\}^T$, the equilibrium solution $\boldsymbol{W}_c^0$ and the transition matrix e^{At} are simplified as[9]

$$\boldsymbol{W}_c^0 = \mathrm{diag}\left[\left(\boldsymbol{W}_c^0\right)_i\right] \tag{28}$$

$$e^{At} = \mathrm{diag}(e^{A_i t}) \tag{29}$$

where

$$\left(\boldsymbol{W}_c^0\right)_i = \begin{bmatrix} (\rho/4\varsigma\omega)_i & 0 \\ 0 & (\rho/4\varsigma\omega)_i \end{bmatrix} \qquad \left(\rho_i = \sum_{k=1}^{m} b_{ki}^2\right) \tag{30}$$

and

$$e^{A_i t} = e^{-\varsigma_i\omega_i t} \times \begin{bmatrix} \dfrac{1}{\omega_{di}}\cos\omega_{di}t + \dfrac{\varsigma_i}{\sqrt{1-\varsigma_i^2}}\sin\omega_{di}t & \dfrac{1}{\sqrt{1-\varsigma_i^2}}\sin\omega_{di}t \\ -\dfrac{1}{\sqrt{1-\varsigma_i^2}}\sin\omega_{di}t & \dfrac{1}{\omega_{di}}\cos\omega_{di}t - \dfrac{\varsigma_i}{\sqrt{1-\varsigma_i^2}}\sin\omega_{di}t \end{bmatrix} \tag{31}$$

Using Eqs. (28) and (29), the controllability grammian matrix $\boldsymbol{W}_c(t_1, t_0)$ in Eq. (27) becomes a block diagonal matrix as follows:

$$\boldsymbol{W}_c(t_1, t_0) = \mathrm{diag}[\boldsymbol{W}_{ci}(t_1, t_0)] \tag{32}$$

where

$$\boldsymbol{W}_{ci}(t_1, t_0) = \rho_i \begin{bmatrix} f_{i1}(t_1, t_0) & f_{i2}(t_1, t_0) \\ f_{i2}(t_1, t_0) & f_{i3}(t_1, t_0) \end{bmatrix} \tag{33}$$

and where f_{ij} is the function of $e^{2\varsigma_i\omega_i t}$, $\cos\omega_{di}t$, and $\sin\omega_{di}t\,(j = 1, 2, 3)$. Therefore, MDOC is reduced to the following simple form:

$$\mathrm{MDOC}_i \equiv \frac{\bar{\boldsymbol{E}}(t_0, t_1; \boldsymbol{\xi}_i)}{\boldsymbol{E}(t_0, t_1; \boldsymbol{\xi}_i)} = \frac{\boldsymbol{\xi}_i^T \bar{\boldsymbol{W}}_c^{-1}(t_1, t_0)\boldsymbol{\xi}_i}{\boldsymbol{\xi}_i^T \boldsymbol{W}_c^{-1}(t_1, t_0)\boldsymbol{\xi}_i}$$

$$= \frac{\rho_i}{\bar{\rho}_i} = \frac{\sum_{k=1}^{m} b_{ki}^2}{\sum_{k=1}^{n_{\mathrm{act}}} b_{ki}^2} \qquad \left(\bar{\rho}_i = \sum_{k=1}^{n_{\mathrm{act}}} b_{ki}^2\right) \tag{34}$$

In the meantime, we can transform MDOE into the following form by using Eqs. (26) and (28):

$$\mathrm{MDOE}_i = \frac{\mathrm{tr}[\boldsymbol{X}_i]}{\mathrm{tr}[\bar{\boldsymbol{X}}_i]} = \frac{-\sum_{k=1}^{m} b_{ki}^2}{-\sum_{k=1}^{n_{\mathrm{excit}}} b_{ki}^2} \tag{35}$$

From Eqs. (34) and (35), we can conclude that, if the natural frequencies of the system are widely spaced and damping is light, MDOC and MDOE of a specific candidate set are identical, even though they are derived from different physical quantities. The optimal candidate sets for actuator and exciter placement will coincide for a single mode.

When we assume the same values for α_i and β_i, then J_c and J_e can be expressed as

$$J_c = \frac{\sum_i^{\mathrm{selected}} (\alpha_i/\rho_i)}{\sum_i^{\mathrm{selected}} (\alpha_i/\bar{\rho}_i)} \qquad J_e = \frac{\sum_i^{\mathrm{selected}} \alpha_i\rho_i}{\sum_i^{\mathrm{selected}} \alpha_i\bar{\rho}_i} \tag{36}$$

From Eq. (36), although cost functions for actuator and exciter placement are defined similarly, they have no relation to each other except for the case when a single mode is of concern. Thus, when two or more modes are of interest, the optimal candidate sets for actuators and exciters may be different.

The cost function J_e for exciter placement prefers the candidate set with a high average of modal energy but does not consider the evenness of the energy distribution to each mode. To give a penalty to a candidate set with uneven energy distribution, the following multiplying factor is included in the definition of the modified cost function J_e' for exciter placement:

$$J_e' = J_e \cdot \min_i \mathrm{MDOE}_i \tag{37}$$

$\min(\mathrm{MDOE}_i)$ in J_e' penalizes the case when the weighted sum of MDOEs is large even though some MDOEs are extremely small. However, this makes J_e' lose its physical meaning as the ratio of the energies.

Numerical Example

The methods derived in the preceding sections will now be illustrated by an example of the two-dimensional frame structure shown in Fig. 1. The model is obtained by the finite element method (FEM) analysis and will serve to confirm the effectiveness of the proposed methods as actuator and exciter placement criteria.

The frame structure is made of aluminum and has 24 nodal points. Each nodal point has two translational modes in the x and y directions and one rotational mode in the z direction, except for nodes 1 and 19, which are fixed to the ground. There are a total of 66 degrees of freedom of the structure. FEM analysis using ANSYS® was carried out, and the first 10 modal frequencies and vectors were computed. The natural frequencies of the frame are summarized in Table 1, and their mode shapes of up to the fifth mode are given in Fig. 2. Because of the boundary condition and the assumption of small deflection, the horizontal and vertical beams can move only in the vertical and horizontal directions, respectively. Possible locations and directions to attach actuators and exciters are given in Table 2.

We shall assume that the modes of interest are those corresponding to the lowest five natural frequencies. If we can use two actuators and exciters for control and system identification, respectively, then the number of possible candidate sets will be $231 (= {}_{22}C_2)$. Table 3 shows the cost functions for the best candidate sets. The α_i in J_c are set to be 1 under the assumption that all modes have the same importance.

We can evaluate the effectiveness of a candidate set for controlling a specific mode based on its MDOCs. For example, the MDOC_1 of

Table 1 Natural frequencies of two-dimensional frame structure

Mode	Frequency, Hz	Mode	Frequency, Hz
1	7.39	6	101
2	26.1	7	116
3	27.4	8	147
4	32.8	9	159
5	75.8	10	188

Table 2 Candidate locations for actuators and exciters

No.	Node point	Direction	No.	Node point	Direction
1	2	$+X$	12	13	$-X$
2	3	$+X$	13	14	$-X$
3	4	$+X$	14	15	$-X$
4	5	$+X$	15	16	$-X$
5	6	$+X$	16	17	$-X$
6	7	$+X$	17	18	$-X$
7	8	$+Y$	18	20	$+Y$
8	9	$+Y$	19	21	$+Y$
9	10	$+Y$	20	22	$+Y$
10	11	$+Y$	21	23	$+Y$
11	12	$+Y$	22	24	$+Y$

Table 3 Performance indices and MDOCs of the optimal candidate sets

Candidate sets	$MDOC_1$	$MDOC_2$	$MDOC_3$	$MDOC_4$	$MDOC_5$	J_c	Rank
$(7X, 9Y)$, $(7X, 11Y)$ $(9Y, 13X)$, $(11Y, 13X)$	0.202	0.103	0.205	0.043	0.139	9.12	1
$(7X, 21Y)$, $(7X, 23Y)$ $(13X, 21Y)$, $(13X, 23Y)$	0.203	0.093	0.040	0.209	0.060	10.91	5
$(5X, 9Y)$, $(5X, 11Y)$ $(9Y, 15X)$, $(11Y, 15X)$	0.087	0.088	0.210	0.044	0.170	11.25	9

Table 4 Performance indices and MDOEs of the optimal candidate sets

Candidate sets	$MDOE_1$	$MDOE_2$	$MDOE_3$	$MDOE_4$	$MDOE_5$	J'_c	Rank
$(7X, 9Y)$, $(7X, 11Y)$ $(9Y, 13X)$, $(11Y, 13X)$	0.202	0.103	0.205	0.043	0.139	11.25	1
$(5X, 9Y)$, $(5X, 11Y)$ $(9Y, 15X)$, $(11Y, 15X)$	0.087	0.088	0.210	0.044	0.170	9.28	5
$(7X, 21Y)$, $(7X, 23Y)$ $(13X, 21Y)$, $(13X, 23Y)$	0.203	0.093	0.040	0.209	0.060	8.56	9

Table 5 Optimal candidate sets for actuators determined by Lim,[4] Hamdan and Nayfeh,[3] and proposed methods

No. of	Lim[4] method		Hamdan and Nayfeh[3] method		Proposed method	
actuators	Candidate sets	α_k^i	Candidate sets	$\Vert f_i \Vert_2$	Candidate sets	MDOC
1	$(22Y)$	1.59	$(22Y)$	8.09	$(22Y)$	0.319
2	$(21Y, 22Y)$, $(22Y, 23Y)$	2.63	$(21Y, 22Y)$, $(22Y, 23Y)$	10.4	$(21Y, 22Y)$, $(22Y, 23Y)$	0.528
3	$(21Y, 22Y, 23Y)$	3.67	$(21Y, 22Y, 23Y)$	12.3	$(21Y, 22Y, 23Y)$	0.738
4	$(10Y, 21Y, 22Y, 23Y)$	4.00	$(10Y, 21Y, 22Y, 23Y)$	12.8	$(10Y, 21Y, 22Y, 23Y)$	0.803
5	$(9Y, 10Y, 21Y, 22Y, 23Y)$	4.22	$(9Y, 10Y, 21Y, 22Y, 23Y)$	13.2	$(9Y, 10Y, 21Y, 22Y, 23Y)$	0.846
	$(10Y, 11Y, 21Y, 22Y, 23Y)$	4.22	$(10Y, 11Y, 21Y, 22Y, 23Y)$	13.2	$(10Y, 11Y, 21Y, 22Y, 23Y)$	0.846

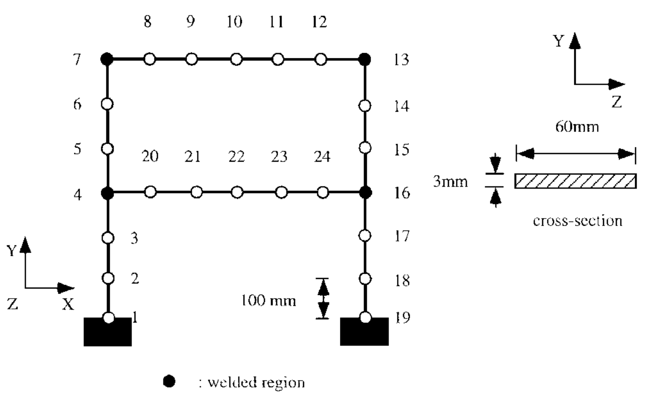


Fig. 1 Two-dimensional frame structure.

candidate set $(7X, 9Y)$ is 0.202, which means that the minimum energy required to control the first mode with this candidate set using 2 actuators is five times as large as the energy needed with the full set of actuators. Thus, the MDOC of one candidate set can be used not only as a comparison parameter but also as an indicator that determines whether the number of actuators used is proper. The optimal candidate sets for controlling modes of interest are $(7X, 9Y)$, $(7X, 11Y)$, $(9Y, 13X)$, and $(11Y, 13X)$ whose values for J_c are 9.12. We also note that the fourth mode will be the most difficult to control with the optimal set of actuators.

The optimal candidate sets for exciters and their MDOEs are shown in Table 4 along with the cost function. The β_i in cost function J'_e are set to be 1. By comparing Table 4 with Table 3, we find that the MDOCs in Table 3 and the MDOEs in Table 4 are identical, as expected. We also note that, although the optimal candidate sets for actuators and exciters coincide, the second optimal candidate sets are different. This can be attributed to the fact that each scheme uses different cost functions.

We have shown how to use the proposed cost functions based on MDOCs and MDOEs to select the optimal sets of actuators and exciters, respectively, when multiple modes are of interest. In this section, we will compare the proposed actuator placement method with existing ones. To facilitate comparison, we consider the case

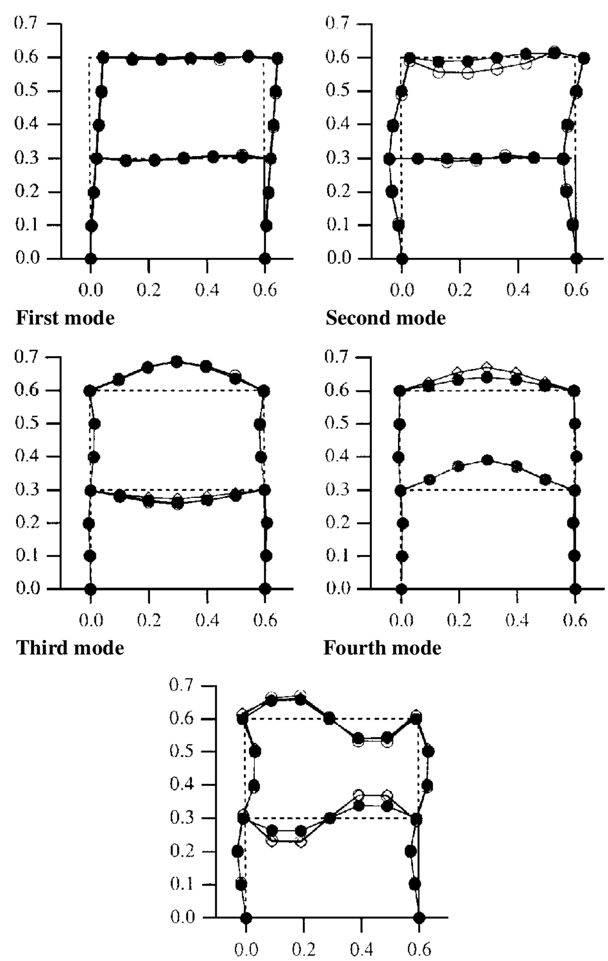


Fig. 2 Mode shapes of frame structure obtained by FEM and modal testing using optimal and poor exciter placement: - - - -, undeformed; ●, FEM; ○, optimal placement; and ◇, poor placement.

where only a single mode is of concern because other placement methods are not available for multiple modes of interest.

The proposed cost function based on a single MDOC is compared with Lim's[4] and Hamdan and Nayfeh's[3] controllability measures to determine the optimal locations for various numbers of actuators. In Lim's[4] method, α_k^i, a parameter based on orthogonal projection of the structural mode into controllable subspace, is used, whereas a 2-norm of f_i, which is a projection of the mode into the range space of the input matrix, is used in Hamdan and Nayfeh's[3] method. Increasing the number of actuators up to five, we have determined the optimal candidate sets of actuators for the fourth mode, which is selected arbitrarily.

We note that the controllability measures given in Table 5 increase as more actuators are used. The optimal candidate sets chosen by the three different measures are shown, and they happen to be identical for every case. By comparing Lim's[4] and Hamdan and Nayfeh's[3] measures with MDOCs, we find that two measures are monotonic with respect to the MDOC. Especially, Lim's measure is proportional to the MDOC. Once again, note that other controllability measures cannot be used systematically when multiple modes are of interest.

Compared with other measures, the proposed MDOC has a bounded value and gives quantitative information about the controllability of the system. An MDOC, which indicates relative control energy requirement, is useful to determine a proper number of actuators. We can make a rational decision on the number of actuators by compromising between the control energy requirement and the equipment cost. In this example, the control system with three optimally located actuators requires about 135% [$\approx(1/0.738)\times 100$] of the energy that is needed for that with the full set of actuators. Three actuators seems to be a reasonable choice because more actuators will not improve the control performance significantly.

Computer Simulation

To verify the appropriateness of the proposed actuator placement criterion, optimal controllers are designed for candidate sets $(7X, 9Y)$ and $(7X, 13X)$, which are an optimal and a poor (not the worst) candidate set, respectively. The worst candidate set $(2X, 18X)$ is not used because it is trivial to compare control performance of two candidate sets when every MDOC of one set is bigger than that of the other set. The poor candidate set $(7X, 13X)$ has the following MDOCs and cost function: MDOC_1, 0.4029; MDOC_2, 0.1849; MDOC_3, 0.1985*e*-09; MDOC_4, 0.1041*e*-08; MDOC_5, 0.5173*e*-02; and J_c, 7.5632*e*-08. It is expected that it will be almost impossible to control modes 3, 4, and 5, with actuators located at the poor candidate set $(7X, 13X)$.

The linear quadratic controller is designed to minimize the following cost function:

$$J = \int_0^\infty (\rho \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x} + \boldsymbol{u}^T \boldsymbol{R} \boldsymbol{u})\, \mathrm{d}t$$

where ρ is the relative weighting between the $\boldsymbol{Q}$ and $\boldsymbol{R}$ matrices. By changing ρ, we can tune the control input energy necessary for control.

Figures 3a and 3b show the root loci of the control system for the two cases, when $\boldsymbol{Q}$ and $\boldsymbol{R}$ are set to be identity matrices and ρ is increased from 0.0 to 1.0. For the optimal actuator candidate set, the poles corresponding to the modes of interest can be moved far into the left-half plane by increasing ρ. This is reasonable because the optimal candidate set $(7X, 9Y)$ is away from any nodal point of the modes of interest (as shown in Fig. 2). But in the case of the poor candidate set $(7X, 13X)$, it is close to nodal points of the third through the sixth modes. Therefore, the poles corresponding to those modes cannot be controlled with any control techniques.

Experiment

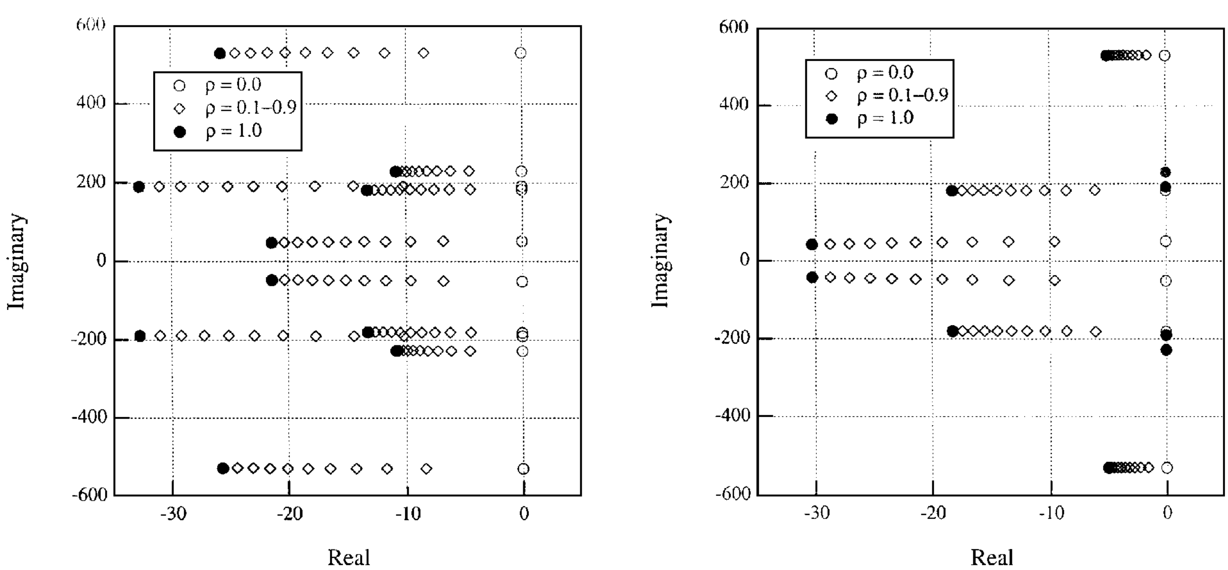
To verify the effectiveness of the proposed exciter placement scheme, we performed modal testing with exciters located at the optimal candidate set $(7X, 11Y)$. For comparison purpose, we repeated the testing using a poor candidate set, $(6X, 10Y)$, whose

Table 6 MACs (%) of modal testing with optimal exciter placement

Mode no.	Frequency, Hz	Mode 1 7.26 (7.39)	Mode 2 25.8 (26.1)	Mode 3 27.5 (27.4)	Mode 4 32.9 (32. 8)	Mode 5 75.8 (75.8)
1	7.26	100				
2	25.8	0.0	100			
3	27.5	0.8	0.9	100		
4	32.9	0.3	0.6	9.7	100	
5	75.8	2.6	0.2	0.0	0.2	100

Table 7 MACs (%) of modal testing with poor exciter placement

Mode no.	Frequency, Hz	Mode 1 7.03 (7.39)	Mode 2	Mode 3 26.4 (27.4)	Mode 4 32.7 (32. 8)	Mode 5 75.7 (75.8)
1	7.03	100				
2	×	×	×			
3	26.4	0.3	×	100		
4	32.7	0.0	×	25.4	100	
5	75.7	5.7	×	0.1	0.0	100



a) Optimal actuator placement **b) Poor actuator placement**

Fig. 3 Root loci of the controller using optimal and poor actuator placement.

MDOEs and cost function are as follows: $MDOE_1$, 0.1472; $MDOE_2$, 0.4881e-04; $MDOE_3$, 0.3048; $MDOE_4$, 0.0657; $MDOE_5$, 0.0468; and J_e', 5.008e-03. $MDOE_2$ is very small because both nodes 6 and 10 are near the nodal points of the second mode. It is expected that we can identify all of the modes of interest with the optimal candidate set, whereas the second mode will be difficult to identify with the poor candidate set.

In Tables 6 and 7, the identified modal frequencies and their modal assurance criterion (MAC) values for both cases are shown; the figures in parentheses represent FEM-based modal frequencies. (See Fig. 2 for the mode shapes identified by the experiments.) As expected, the experiment using optimal exciter placement has identified all of the modes (first through fifth mode), and their orthogonality seems good considering the off-diagonal terms of the MAC table. However, in the experiment using poor placement, we could not identify the second structural mode, and elements of MAC corresponding to the second mode are missing. Also, note that the estimation error of the natural frequencies for the case with optimal placement is smaller than that of the case with poor placement.

Concluding Remarks

A methodology is proposed for optimal actuator placement for the control of flexible structures based on the MDOCs. The MDOC indicates the relative ratio of the minimum energies required to regulate a system from a modal disturbance when a specific candidate set and the full set of actuators are used, respectively. In the case that structural modes of interest are determined through a priori analysis, we can find the optimal candidate sets for the specific number of actuators and also determine whether the number of actuators used is sufficient, based on the cost function and MDOCs of those modes.

The exciter placement scheme in system identification was also investigated. An MDOE that is similar to the MDOC was defined that reflects the steady-state energy distribution in each mode. Under an assumption of widely spaced natural frequencies and light damping, the MDOC and the MDOE are shown to have the same values for a given mode, implying that the optimal candidate sets for actuators and exciters will coincide if a single mode is of interest. However, if several structural modes are of interest, the placement criteria for actuators and exciters become different and the resulting optimal candidate sets will turn out to be different.

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